The asymptotic expansions at large Reynolds numbers for steady motion between non-coaxial rotating cylinders

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(*Received* 7 *May* 1957)

SUMMARY

The motion of a viscous fluid contained between two rotating, circular cylinders whose axes are set slightly apart is considered. The equations of viscous motion are linearized by expanding the stream function in the form $\sum_{n=0}^{\infty} \psi_n \gamma^n$, where γ is a parameter which depends on the distance between the cylinder axes. The ensuing analysis appears to hold for all values of the fluid viscosity ν, and in particular for small values of ν.

The asymptotic behaviour of the solutions for small ν is examined, attention being mainly confined to the first order stream function ψ_1 and the corresponding component of vorticity ζ_1. Outside the boundary layers, where, for small ν, ψ_1 may be expanded asymptotically as $\sum_{n=0}^{\infty} \psi_1^{(n)} \nu^{n/2}$, the terms $\zeta_1^{(n)}$ of the corresponding expansions for the vorticity are shown to be uniform throughout the fluid. It is noted that the asymptotic expansions of ψ_1 for the region of the boundary layers and for the region outside the boundary layers may be combined in a single expansion which holds in both regions. The leading terms of this expansion are calculated by boundary layer methods.

1. INTRODUCTION

This paper is concerned with a viscous motion whose governing equations may, in principle, be solved exactly. The asymptotic behaviour of the solution when the fluid viscosity ν is small is derived and discussed and a procedure formulated by which the higher order corrections to the boundary layer approximation may be determined. Inasmuch as the streamlines of the motion are closed the results bear particularly on closed flows.

Now, the boundary layer approximation, as conceived by Prandtl, determines the stream function ψ of a steady two-dimensional motion with an error which is, in general, $O(UL\{\nu/UL\})$ in the boundary layer and $O(UL\{\nu/UL\}^{1/2})$ elsewhere, U and L being respectively a typical speed and a typical length of the motion. For certain configurations even this accuracy cannot be attained. If for example the fluid impinges on a sharp edge as in the uniform streaming past a flat plate, the Prandtl approximation fails to determine the stream function correctly near the edge. More serious difficulties attend the shedding of a boundary layer to form a wake. Such configurations aside, the problem remains of how best to determine the higher order corrections to the original approximation.

One correction can be made at once; that is, the displacement of the streamlines outside the boundary layer caused by the slow movement of fluid inside the layer can be calculated. In a recent paper, Kaplun (1954) has shown that by suitable choice of coordinates the solution of the boundary layer approximation can be made to hold throughout the whole region of flow and made to include, outside the boundary layer, the effects of displacement thickness.

In order to formulate a procedure for determining the higher order corrections, it is helpful to have as a guide one or more exact solutions to the equations of motion, whose asymptotic behaviour for small ν has been elucidated. One such solution, though for an unsteady motion, has been analysed by Lagerstrom & Cole (1955). Lagerstrom & Cole consider the motion set up by a uniformly expanding circular cylinder moving parallel to its axis in unbounded fluid. The exact solutions obtained by Lagerstrom & Cole are of the form

$$\psi \sim \sum_{n=0}^{m} \mathscr{R}_n(\mathbf{r}, t, \nu)\nu^{-n/2} + o(\nu^{-m/2}),$$

where the coefficients $\mathscr{R}_n$ are significant in the boundary layer and elsewhere are transcendentally small, and they propose, from general considerations (which are not stated), that the stream function of steady motion with no wake may be generally represented in the form

$$\psi \sim \sum_{n=0}^{m} \mathscr{I}_n(\mathbf{r})\nu^{-n/2} + \sum_{n=0}^{m} \mathscr{R}_n(\mathbf{r}, \nu)\nu^{-n/2} + o(\nu^{-m/2}),$$

where the first term represents the truncated asymptotic series for ψ at a fixed point $\mathbf{r}$.

The example considered in the following concerns the motion of fluid between two rotating, circular cylinders whose axes are parallel and set slightly apart. When the separation of the cylinder axes is small compared with the difference in length of their radii the equations of motion may be solved by a perturbation method. Moreover the perturbation expansions appear to hold uniformly for all ν, providing that the expansion parameter is sufficiently small. The solution therefore determines the asymptotic behaviour of the motion when ν is small.

2. Linearization of the equations of motion

We begin by defining the perturbation expansions and deriving equations of motion for the first order terms.

Let the radii of the two cylinders be a, b and the distance between their axes $a\epsilon$. Take polar coordinates r, θ in any cross-section, with origin at the axis of the inner cylinder and reference line $\theta = 0$ along the radius to the axis of the outer cylinder (see figure 1). Introduce further the non-dimensional coordinates ρ, ϕ defined in terms of r, θ by

$$\left.\begin{aligned} z &= a\,\frac{\zeta+\gamma}{1+\gamma\zeta}, \\ z &= re^{i\theta}, \qquad \zeta = \rho e^{i\phi}, \end{aligned}\right\} \tag{1}$$

with

$$\gamma = -2\epsilon[(b/a)^2-1-\epsilon^2+\sqrt{\{(b^2/a^2-1-\epsilon^2)^2-4\epsilon^2\}}]^{-1}. \tag{2}$$

The circular sections of the inner and outer cylinders are then the coordinate lines $\rho = 1$ and $\rho = \beta$, where

$$\beta = \frac{(b/a)+\epsilon-\gamma}{1-(b/a)\gamma-\epsilon\gamma}. \tag{3}$$

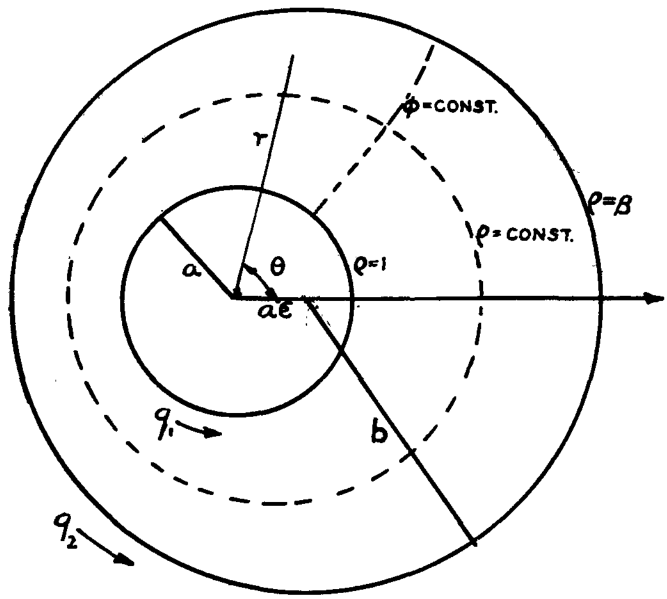

Figure 1.

The remaining coordinate lines ρ = constant and ϕ = constant constitute the two orthogonal pencils of circles generated by the circular cylinder sections The coordinate mesh is clearly the same as that of the two-dimensional bipolar system in which the cylinder sections are coordinate lines. The coordinate variables ρ, ϕ are however different, having been modified so as to have the advantage in this problem of degenerating to polar variables r, θ when the cylinders are coaxial ($\epsilon = \gamma = 0$).

The rotary motion between cylinders is supposed two-dimensional and may therefore be represented in terms of a non-dimensional stream function ψ, the velocity components in the coordinate directions ρ increasing, ϕ increasing, being respectively

$$u_\rho = q_1\frac{\sqrt{J}}{\rho}\frac{\partial\psi}{\partial\phi}, \qquad u_\phi = -q_1\sqrt{J}\,\frac{\partial\psi}{\partial\rho}, \tag{4}$$

where q_1 is the peripheral speed of the inner cylinder. The function J is in effect the Jacobian of the transformation (1) and is defined by

$$J = \frac{(1+2\gamma\rho\cos\phi+\gamma^2\rho^2)^2}{(1-\gamma^2)^2}. \tag{5}$$

On eliminating the pressure, the equations of motion reduce to

$$-\frac{1}{\rho}\frac{\partial(\psi,\zeta)}{\partial(\rho,\phi)} = \frac{1}{R}\nabla^2\zeta, \qquad \zeta = -J\nabla^2\psi, \tag{6}$$

where ζ is the axial component of vorticity, R is the Reynolds number, defined here by aq_1/ν, and, in both equations,

$$\nabla^2 \equiv \frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2}. \tag{7}$$

(In using q_1 as a typical speed, it has been tacitly assumed that the inner cylinder is not at rest. If it were at rest, the typical speed could with only trivial changes be taken to be the peripheral speed q_2 of the outer cylinder.) The boundary conditions at the inner and outer cylinders are respectively,

$$\left.\begin{aligned} &\psi(1,\phi) = 0, && \frac{\partial}{\partial\rho}\psi(1,\phi) = -\frac{1}{\sqrt{\{J(1,\phi)\}}}, \\ &\psi(\beta,\phi) = \text{constant}, && \frac{\partial}{\partial\rho}\psi(\beta,\phi) = -\frac{q_2}{q_1\sqrt{\{J(\beta,\phi)\}}}. \end{aligned}\right\} \tag{8}$$

Further the solution to (6) must be such that the pressure is single-valued. These equations and boundary conditions suffice to determine ψ. From them it is clear that ψ depends on the geometrical parameters β, γ, the ratio of the peripheral speeds q_1/q_2, and the Reynolds number R.

It is now proposed to consider the motion when the separation $a\epsilon$ of the cylinder axes is small compared with the difference in length $b-a$ of their radii (that is, $\gamma \ll 1$). In the particular instance where the cylinders are coaxial the motion is axi-symmetric and independent of the Reynolds number. The stream function is then

$$\psi_0(\rho,\beta,q_2/q_1) = -\tfrac{1}{2}A\rho^2 - B\log\rho, \tag{9}$$

where

$$A = \frac{\beta(q_2/q_1)-1}{\beta^2-1}, \qquad B = \frac{\beta^2-\beta(q_2/q_1)}{\beta^2-1}; \tag{10}$$

and the vorticity is

$$\zeta_0(\beta,q_2/q_1) = 2A. \tag{11}$$

In general, when the cylinders are nearly coaxial, it is assumed that the stream function ψ and the vorticity component ζ may be expanded in the forms

$$\left.\begin{aligned} \psi &= \psi_0(\rho,\beta,q_2/q_1) + \sum_{n=1}^{\infty}\psi_n(\rho,\phi,\beta,q_2/q_1,R)\gamma^n, \\ (1-\gamma^2)^2\zeta &= 2A(\beta,q_2/q_1) + \sum_{n=1}^{\infty}\zeta_n(\rho,\phi,\beta,q_2/q_1,R)\gamma^n. \end{aligned}\right\} \tag{12}$$

It is the first order motion described by the first order terms with which this paper is principally concerned. The higher order terms are discussed briefly in the final section.

Concerning the mode of expansion, note that the coordinate description of the bounding surfaces does not involve the expansion parameter γ. This is important if the expansion is to be used for large Reynolds numbers, and is the reason for introducing 'bipolar' coordinates. If the coordinate description of the bounding surfaces involved γ then the boundary conditions would have to be applied at surfaces $\gamma = 0$ different from the cylinders. For this purpose, the velocity field at each boundary would have to be expanded in powers of 'distance' from the surface $\gamma = 0$ to the boundary, and it is unlikely that such series would converge when R is large. Note also that when $\gamma \neq 0$, the motion described by $\psi_0(\rho, q_2/q_1, \beta)$ is not axi-symmetric. In this motion the circles ρ = constant and, in particular, the cylinders are streamlines, the speeds at the inner and outer cylinders being respectively $q_1\sqrt{\{J(1,\phi,\gamma)\}}$, $q_2\sqrt{\{J(\beta,\phi,\gamma)\}}$.

When the expansions of ψ, ζ and the analogous expansion of J are substituted into the equations of motion (6) and the boundary conditions (8), and the leading terms isolated, we get

$$\left(A+\frac{B}{\rho^2}\right)\frac{\partial \zeta_1}{\partial \phi} = \frac{1}{R}\nabla^2\zeta_1, \tag{13 a}$$

$$\zeta_1 = -\nabla^2(\psi_1 - A\rho^3\cos\phi), \tag{13 b}$$

with the boundary conditions on ψ_1:

$$\left.\begin{aligned} &\psi_1(1,\phi) = 0, && \frac{\partial}{\partial\rho}\psi_1(1,\phi) = 2\cos\phi, \\ &\psi_1(\beta,\phi) = \text{constant}, && \frac{\partial}{\partial\rho}\psi_1(\beta,\phi) = 2\,q_2/q_1\,\beta\cos\phi. \end{aligned}\right\} \tag{14}$$

Further, from the corresponding exact condition, the solution of (13) must be such that the first order perturbation in pressure is single-valued.

The equations governing the first order motion contain the Reynolds number in the same way as the exact equations but have the merit of being linear and hence more tractable. In that the convective velocity in the vorticity equation (13) is known, this equation is mathematically similar to Oseen's equation and the generalisation of it discussed by Zeilon (1927) and Burgers (1921). In the present case, there is, however, reason to believe that the perturbation to ψ described by the equations (13) is uniformly valid to $O(\gamma)$ even at large Reynolds numbers. A similar linearization of the equations of motion was discussed by Proudman (1956) who investigated the flow between two concentric spheres which rotate with slightly different angular velocities. In the simpler configuration considered here the linearized equations may be solved exactly. This we proceed to do.

3. The first order solution

It is evident from (13) and (14), that ψ_1 and ζ_1 vary with ϕ according to the simple forms

$$\left.\begin{aligned} \psi_1 &= A\rho^3\cos\phi + 2\mathscr{R}\{f(\rho)e^{i\phi}\}, \\ \zeta_1 &= 2\mathscr{R}\{g(\rho)e^{i\phi}\}. \end{aligned}\right\} \tag{15}$$

The way in which ψ_1 and ζ_1 vary with ρ is then determined from the differential equations

$$\left.\begin{aligned} g''+\frac{g'}{\rho}-\left(RAi+\frac{RBi+1}{\rho^2}\right)g=0,\\ f''+\frac{f'}{\rho}-\frac{f}{\rho^2}=-g,\end{aligned}\right\} \tag{16}$$

with the boundary conditions

$$\left.\begin{aligned} f(1)&=-\tfrac{1}{2}A, & f'(1)&=1-\tfrac{3}{2}A,\\ f(\beta)&=-\tfrac{1}{2}A\beta^2, & f'(\beta)&=\frac{q_2}{q_1}\beta-\tfrac{3}{2}A\beta^2,\end{aligned}\right\} \tag{17}$$

where primes denote differentiation with respect to ρ.

Provided that the vorticity of the motion described by ψ_0 is not zero $(A\neq 0)$, $g(\rho)$ satisfies a Bessel's equation of order $\mu=\sqrt{(BRi+1)}$ with independent variable $z=\sqrt{(-ARi)}\rho$. Thus

$$g(\rho)=-2CJ_\mu(z)-2DH^1_\mu(z), \tag{18}$$

where $J_\mu(z)$, $H^1_\mu(z)$ denote respectively the Bessel function and Hankel function of order μ, and C, D are constants to be determined. For definiteness we take $|\arg\mu|<\frac{1}{2}\pi$ and $|\arg z|=\frac{1}{4}\pi$. A general integral of (16) may now be seen to be

$$f(\rho)=A_1\rho+\frac{B_1}{\rho}+CI_1(\rho)+DI_2(\rho), \tag{19}$$

where

$$\left.\begin{aligned} I_1(\rho)&=\rho\int_1^\rho J_\mu(z)\,d\rho-\frac{1}{\rho}\int_1^\rho \rho^2 J_\mu(z)\,d\rho,\\ I_2(\rho)&=\rho\int_\beta^\rho H^1_\mu(z)\,d\rho-\frac{1}{\rho}\int_\beta^\rho \rho^2 H^1_\mu(z)\,d\rho.\end{aligned}\right\} \tag{20}$$

The constants A_1, B_1, C, D may be evaluated from the boundary conditions (17). Whence

$$\left.\begin{aligned} A_1&=\tfrac{1}{2}-A-\tfrac{1}{2}D\{I_2(1)+I_2'(1)\},\\ B_1&=-\tfrac{1}{2}B-\tfrac{1}{2}D\{I_2(1)-I_2'(1)\},\\ \Delta C&=\left(\beta+\frac{1}{\beta}-\frac{2q_2}{q_1}\right)I_2(1)+\left(\beta-\frac{1}{\beta}\right)I_2'(1),\\ \Delta D&=\left(2-\beta-\frac{1}{\beta}\right)I_1(\beta)+(\beta^2-1)\frac{q_2}{q_1}I_2'(\beta),\end{aligned}\right\} \tag{21}$$

where

$$\Delta=\left(1-\frac{1}{\beta^2}\right)\{\beta I_1'(\rho)I_2'(1)-I_1(\beta)I_2(1)\}+ \\ +\left(1+\frac{1}{\beta^2}\right)\{\beta I_1'(\beta)I_2(1)-I_1(\beta)I_2'(1)\}. \tag{22}$$

Thus, when the vorticity of the unperturbed motion is non-zero ($A \neq 0$), the solution for ψ_1 is

$$\psi_1 = A\rho^3 \cos\phi + 2\mathscr{R}\left\{\left[A_1\rho + \frac{B_1}{\rho} + CI_1(\rho) + DI_2(\rho)\right]e^{i\phi}\right\}, \tag{23}$$

where I_1, I_2, A_1, B_1, C, D are defined in (20) to (22); whilst the solution for ζ_1 is

$$\zeta_1 = -4\mathscr{R}\{[CJ_\mu(z) + DH^1_\mu(z)]e^{i\phi}\}. \tag{24}$$

The exceptional case in which the motion described by ψ_0 is irrotational ($A = 0$) arises when $q_2/q_1 = 1/\beta$. In this case $g(\rho)$ has the form

$$g(\rho) = -2C\rho^\mu - 2D\rho^{-\mu}. \tag{25}$$

The function $f(\rho)$ may be obtained in the same way as when $A \neq 0$, the 'Bessel' functions $J_\mu(z)$, $H^1_\mu(z)$ being replaced by ρ^μ, $\rho^{-\mu}$. Thus

$$f(\rho) = A_1\,\rho + \frac{B_1}{\rho} + CI_1(\rho) + DI_2(\rho), \tag{26}$$

where I_1, I_2 are now

$$\left.\begin{aligned} I_1(\rho) &= \frac{\rho}{1+\mu}(\rho^{1+\mu} - 1) - \frac{1}{(3+\mu)\rho}(\rho^{3+\mu} - 1), \\ I_2(\rho) &= \frac{\rho}{1-\mu}(\rho^{1-\mu} - \beta^{1-\mu}) - \frac{1}{(3-\mu)\rho}(\rho^{3-\mu} - \beta^{3-\mu}). \end{aligned}\right\} \tag{27}$$

The constants A_1, B_1, C, D are again related to I_1 and I_2 as in (21). The solution for ψ_1 is therefore formally the same as in (23), the leading term now vanishing and the functions $I_1(\rho)$, $I_2(\rho)$ being defined as in (27). The solution for ζ_1, however, changes to

$$\zeta_1 = -4\mathscr{R}\{[C\rho^u + D\rho^{-\mu}]e^{i\phi}\}. \tag{28}$$

The significance of these solutions is that they express in closed form the dependence of a viscous motion on Reynolds number. In what follows they are used to derive the asymptotic behaviour of the motion when the Reynolds number is large. Certain properties of this motion may of course be anticipated. Thus we may expect viscous action to be negligible save in thin layers at the cylinders, and in these layers the Prandtl approximation should hold. Further, the vorticity at infinite Reynolds number should be uniform, save at the cylinders (Batchelor 1956). Our main interest in deriving the asymptotic behaviour of ψ_1, however, lies in the higher order corrections to the Prandtl approximation, or, equally, in the higher order terms of the asymptotic representations of ψ_1.

4. Behaviour of the solution to the first order vorticity equation for large R

As a preliminary to deriving the behaviour of the first order stream function ψ_1 for large R, it is helpful to consider first the behaviour of the vorticity component ζ_1. It is sufficient, for this purpose, to confine attention to those properties that follow directly from the vorticity equation (13 a), when the way in which the vorticity component ζ_1 varies periodically with ϕ is known. In view of the similarity of the approximate vorticity equation to the exact vorticity equation expressed in coordinates for which the stream-lines are coordinate lines, these properties may be of interest in suggesting general properties of the vorticity distribution for motions with closed stream-lines.

Before proceeding further, it should be noted that the method of expanding the stream function as a series in γ cannot be expected to hold for large R if the outer cylinder is at rest. The reason for this may be seen by referring to the boundary layer approximation. If the series were valid and a boundary layer existed on the outer cylinder, the speed of the inviscid flow at the outer cylinder would be expected to be $O(\gamma q_1)$. The thickness of the boundary layer there would then be $O(a\gamma^{-1/2} R^{-1/2})$. Thus for a given value of γ and a Reynolds number much greater than $1/\gamma$ (a necessary condition for the boundary layer to exist), the flow quantities of the boundary layer would not be analytic functions of γ. For instance, the velocity gradient $\partial u_\phi/\partial \rho$ at the outer cylinder would behave like

$$q_1 \gamma^{3/2} R^{1/2} \times (\text{a function of } \phi, \text{ independent of } \gamma \text{ and } R).$$

A similar objection would apply if the inner cylinder were at rest. From here on it will therefore be assumed that neither cylinder is at rest.

It will also be supposed that the cylinders rotate in the same sense. With this provision, the speed of the unperturbed flow does not vanish at any point of the fluid and some simplification thereby results in the behaviour of the vorticity at large overall Reynolds numbers.

The general solution to the first order vorticity equation for which ζ_1 has the appropriate dependence of ϕ is given by (24) or, if $A = 0$, by (28). In both equations C, D are for the present to be regarded as undetermined constants. The behaviour of ζ_1 for large Reynolds numbers is thus governed (C, D apart) by the symptotic behaviour of $J_\mu(z)$ and $H^1_\mu(z)$, or, if $A = 0$, by the asymptotic behaviour of ρ^μ and $\rho^{-\mu}$.

When the cylinder speeds are such that $A \neq 0$ and $B \neq 0$, the behaviour of $J_\mu(z)$, $H^1_\mu(z)$ when both order and argument are large is required. With some minor modification due to substituting for the variables μ and z the more readily interpreted variables ρ, A, B and R, the asymptotic representations obtained by Debye may be used. An auxiliary variable σ is introduced which is related to the local speed

$$q_1 Q(\rho) = q_1\left(A\rho + \frac{B}{\rho}\right)$$

by

$$\sigma = \int^\rho \left(\frac{Q}{\rho}\right)^{1/2} d\rho. \tag{29}$$

The asymptotic expansions of $J_\mu(z)$, $H_\mu^1(z)$ for the values which z/μ assume in this problem may then be written (see Watson 1944, also Olver 1954)

$$\left.\begin{aligned} J_\mu(z) &\sim k_1\, e^{\sqrt{(Ri)}\sigma} P(R^{1/2}), \\ H_\mu^1(z) &\sim k_2\, e^{-\sqrt{(Ri)}\sigma} P(-R^{1/2}), \end{aligned}\right\} \tag{30}$$

where k_1, k_2 are constants independent of ρ, whose values need not be known here, and $P(R^{1/2})$ is an asymptotic series in $R^{1/2}$ whose coefficients are regular functions of ρ, A, B.

If the cylinder speeds are such that in the unperturbed motion $A = 0$ or $B = 0$, the behaviour of the vorticity component depends on the behaviour of $\rho^{\sqrt{(Ri+1)}}$, $\rho^{-\sqrt{(Ri+1)}}$ (see (28)) or of $J_1(\sqrt{(-Ri)}\rho)$, $H_1^1(\sqrt{(-Ri)}\rho)$. The asymptotic representations of these functions are analytically simpler than the asymptotic representations of $J_\mu(z)$, $H_\mu^1(z)$ when $A \neq 0$, $B \neq 0$ but are not otherwise importantly different. In fact, when $A = 0$ the asymptotic behaviour of $\rho^{\sqrt{(Ri+1)}}$, $\rho^{-\sqrt{(Ri+1)}}$ is correctly represented by the respective expressions for $J_\mu(z)$, $H_\mu^1(z)$ set out in (30) providing that $\log\rho$ is substituted for σ. Likewise, when $B = 0$, the asymptotic behaviour of $J_1(\sqrt{(-Ri)}\rho)$, $H_1^1(\sqrt{(-Ri)}\rho)$ is correctly represented by the above expressions for $J_\mu(z)$, $H_\mu^1(z)$ providing that ρ is substituted for σ. In either case, the function adopted for $\sigma(\rho)$ is related to the local speed of the unperturbed flow as in (29).

Thus, when the cylinders rotate in the same sense, the behaviour of the first order vorticity component at large Reynolds number may in every case be represented in the same asymptotic form : namely,

$$\zeta_1 \sim \mathscr{R}\{\kappa_1\, P(R^{1/2}) e^{\sqrt{(Ri)}\sigma + i\phi} + \kappa_2\, P(-R^{1/2}) e^{-\sqrt{(Ri)}\sigma + i\phi}\}, \tag{31}$$

where κ_1, κ_2 are constants independent of position, $P(R^{1/2})$ denotes an asymptotic series in $R^{1/2}$,and $\sigma(\rho)$ is related to the speed of the unperturbed motion as in (29).

It follows immediately from the way in which σ is related to the speed Q of the unperturbed motion, that $\sigma(\rho)$ is monotonic increasing; the complication that might have arisen if the speed of the unperturbed motion had vanished at some circle $\rho =$ constant between the cylinders having been avoided by supposing that both cylinders rotate in the same sense. The vorticity component ζ_1 thus resolves into two parts, both of which oscillate rapidly with changes in ρ, the amplitude of the first part decaying exponentially with increasing distance from the outer cylinder and the amplitude of the second part decaying exponentially with increasing distance from the inner cylinder. This 'exponential' property of the two parts that contribute to ζ_1 will be seen to play a significant role in determining the behaviour of the motion at large Reynolds number.

5. Behaviour of the first order stream function for large R

We now consider the behaviour for large R of the first order stream function ψ_1. It is to be anticipated that the behaviour of ψ_1 outside the boundary layers at each cylinder will differ from its behaviour inside these layers. Accordingly the asymptotic behaviour of ψ_1 is derived firstly for points in the fluid whose distance from the nearest boundary is very much greater than $a/R^{-1/2}$ and secondly for points whose distance from the nearest boundary is $O(a/R^{-1/2})$.

It is convenient to resolve the solution for ψ_1 into the components

$$\psi_R = 2\mathscr{R}\{(CI_1 + DI_2)e^{i\phi}\},$$

$$\psi_I = A\rho^3 \cos\phi + 2\mathscr{R}\left\{\left(A_1\rho + \frac{B_1}{\rho}\right)e^{i\phi}\right\}.$$

The component ψ_R gives rise to the vorticity distribution ζ_1 and will be referred to as the strongly rotational component, whilst ψ_I makes no contribution to ζ_1 and will be referred to as the weakly rotational component. The strongly rotational component may further be divided into two parts, namely

$$\psi_{R1} = 2\mathscr{R}\{CI_1 e^{i\phi}\}, \qquad \psi_{R2} = 2\mathscr{R}\{DI_2 e^{i\phi}\},$$

each corresponding to one of the two parts of ζ_1 distinguished above.

For the case where the cylinder speeds are such that $A \neq 0$, the strongly rotational terms ψ_{R1}, ψ_{R2} give rise to the respective vorticity distributions $2\mathscr{R}\{CJ_\mu(z)e^{i\phi}\}$, $2\mathscr{R}\{DH^1_\mu(z)e^{i\phi}\}$. The functions I_1 and I_2 which define the way in which ψ_{R1}, ψ_{R2} vary with ρ, are expressed in terms of simple integrals of the corresponding functions $J_\mu(z)$, $H^1_\mu(z)$, and their asymptotic behaviour is therefore related to that of $J_\mu(z)$, $H^1_\mu(z)$. The detailed definitions of I_1, I_2 were given in (20). Whence also

$$\left.\begin{aligned} I_1' &= \int_1^\rho J_\mu(z)\, d\rho + \frac{1}{\rho^2}\int_1^\rho \rho^2 J_\mu(z)\, d\rho, \\ I_2' &= \int_\beta^\rho H^1_\mu(z)\, d\rho + \frac{1}{\rho^2}\int_\beta^\rho \rho^2 H^1_\mu(z)\, d\rho. \end{aligned}\right\} \tag{32}$$

On substituting for $J_\mu(z)$ and $H^1_\mu(z)$ the asymptotic expansions (30) we find that

$$\left.\begin{aligned} I_1 &\sim \kappa_1\left\{a_1 R^{-1/2} + \sum_{n=2}^{\infty} a_n R^{-n/2}\right\}e^{\sqrt{(Ri)}\sigma}, \\ I_2 &\sim \kappa_2\left\{b_1 R^{-1/2} + \sum_{n=2}^{\infty} b_n R^{-n/2}\right\}e^{-\sqrt{(Ri)}\sigma}, \\ I_1' &\sim \kappa_1\left\{\sqrt{(i)}\sigma' a_1 + \sum_{n=2}^{\infty} a_n^* R^{-n/2}\right\}e^{\sqrt{(Ri)}\sigma}, \\ I_2' &\sim \kappa_2\left\{-\sqrt{(i)}\sigma' b_1 + \sum_{n=2}^{\infty} b_n^* R^{-n/2}\right\}e^{-\sqrt{(Ri)}\sigma}, \end{aligned}\right\} \tag{33}$$

where the coefficients a_n, a_n^*, b_n, b_n^* are independent of R. As was to be expected, in view of the corresponding behaviour of $J_\mu(z)$, both I_1 and I_1' are smaller at any point ρ than at the outer cylinder ($\rho = \beta$) by a factor of $O(e^{-\sqrt{(\frac{1}{2}R)}d_2})$ where $d_2 = \sigma(\beta) - \sigma(\rho) > 0$. Similarly, I_2 and I_2' are both smaller at any point ρ than at the inner cylinder ($\rho = 1$) by a factor of $O(e^{-\sqrt{(\frac{1}{2}R)}d_1})$, where $d_1 = \sigma(\rho) - \sigma(1) > 0$. The asymptotic representations of ψ_{R1}, ψ_{R2} and also

ψ_I now follow from (33) and the expressions (21), (22) for the constants A, B, C, D. Hence,

$$\psi_{R1} \sim \frac{2}{R^{1/2}} \mathscr{R}\left\{\left[\sqrt{\left(\frac{\beta q_1}{iq_2}\right)} + \sum_{n=1}^{\infty} \alpha_n R^{-n/2}\right] e^{-\sqrt{(Ri)}\{\sigma(\beta)-\sigma(\rho)\}+i\phi}\right\}, \tag{34 a}$$

$$\psi_{R2} \sim \frac{2}{R^{1/2}} \mathscr{R}\left\{\left[-\frac{\beta q_2}{q_1\sqrt{i}} + \sum_{n=1}^{\infty} \beta_n R^{-n/2}\right] e^{-\sqrt{(Ri)}\{\sigma(\rho)-\sigma(1)\}+i\phi}\right\}, \tag{34 b}$$

$$\psi_I \sim -\frac{A}{\rho}(\rho^2-1)(\beta^2-\rho^2)\cos\phi + \frac{2}{R^{1/2}} \mathscr{R}\left\{\left(\frac{\beta}{(\beta^2-1)\sqrt{i}}\left[\left\{\frac{\beta^2 q_2}{q_1} + \sqrt{\left(\frac{\beta q_1}{q_2}\right)}\right\}\frac{1}{\rho} - \left\{\frac{q_2}{q_1} + \sqrt{\left(\frac{\beta q_1}{q_2}\right)}\right\}\rho\right] + \sum_{n=2}^{\infty} (l_n \rho + m_n \rho^{-1}) R^{-n/2}\right) e^{i\phi}\right\}, \tag{34 c}$$

where α_n, β_n are independent of R, and l_n, m_n are independent of R and ρ. It is evident from (34), that the constants A, B, C, D are such that the strongly rotational components ψ_{R1}, ψ_{R2} and the weakly rotational component ψ_I are each $O(R^{-1/2})$ at the bounding cylinders. The fact that the components ψ_{R1}, ψ_{R2} are transcendentally small save in thin layers of thickness $O(aR^{-1/2})$ at the inner and outer cylinders follows automatically from the ' exponential ' behaviour of these components.

In the special case where $A = 0$ the analysis may be retraced with only small changes, caused by replacing $J_\mu(z)$, $H^1_\mu(z)$ by ρ^μ, $\rho^{-\mu}$; and, on putting $\log \rho$ for σ, the results (34) are obtained unimpaired. The asymptotic representations (43) therefore hold for all relevant values of A.

The asymptotic representation of the first order stream function ψ_1 appropriate to points whose distance from either boundary is much greater than $aR^{-1/2}$ is evidently that given by (34) for ψ_I. For points whose distance from either cylinder is $O(aR^{-1/2})$, the strongly rotational component is significant. Consider the flow near the inner cylinder. The asymptotic representation of ψ_1 for a point $\rho = 1 + \xi R^{-1/2}$, where ξ is independent of Reynolds number, clearly includes ψ_I and ψ_{R2} but not ψ_{R1}. Whence, from (34 b) and (34 c), we find after some rearrangement that for fixed ξ,

$$\psi_1 \sim \frac{2}{R^{1/2}} \mathscr{R}\left\{\left[\left(1-\frac{\beta q_2}{q_1}\right)\xi + \frac{\beta q_2}{q_1\sqrt{i}}(1-e^{-\sqrt{i}\xi})\right] e^{i\phi}\right\} + \sum_{n=2}^{\infty} \mathscr{R}\{P_n e^{i\phi}\} R^{-n/2} + \sum_{n=2}^{\infty} \mathscr{R}\{Q_n e^{i\phi-\sqrt{i}\xi}\} R^{-n/2}, \tag{35}$$

where P_n and Q_n denote polynomials in ξ of orders n and $n-1$ respectively. In the same way the asymptotic representation of ψ_1 for a point $\rho = \beta - \eta R^{-1/2}$, where η is independent of R, near the outer cylinder includes ψ_1 and ψ_{R1} but not ψ_{R2}. Thus, from (34 a) and (34 c) we find that for fixed η

$$\psi_1 \sim \frac{2}{R^{1/2}} \mathscr{R}\left\{\left[\left(1-\frac{\beta q_2}{q_1}\right)\eta - \sqrt{\left(\frac{\beta q_1}{iq_2}\right)}(1-e^{-\sqrt{(iq_2/\beta q_1)}\eta})\right] e^{i\phi}\right\} + \sum_{n=2}^{\infty} \mathscr{R}\{S_n e^{i\phi}\} R^{-n/2} + \sum_{n=2}^{\infty} \mathscr{R}\{T_n e^{i\phi-\sqrt{(iq_2/\beta q_1)}\eta}\} R^{-n/2}, \tag{36}$$

where S_n and T_n denote polynomials in η (different from before) of orders n and $n-1$ respectively.

The main features of the motion described by these expansions are twofold. The first is that the stream function ψ_1 may be expressed both outside and, with suitably rescaled coordinates, inside the boundary layers as an asymptotic series in $R^{1/2}$. The zero order stream function ψ_0 is independent of R, so that similar expansions hold for $\psi_0+\gamma\psi_1$. The second feature is that the vorticity ζ_1 associated with ψ_1 is zero, to the accuracy of the asymptotic representation for ψ_1, in the region outside the boundary layers. The vorticity ζ_0 is constant, so that the vorticity $\zeta_0+\gamma\zeta_1$ associated with $\psi_0+\gamma\psi_1$ is again, to the accuracy of the asymptotic representation, uniform throughout the region outside the boundary layers. The leading terms of the expansions (34 c), (35), (36) for ψ_1 are those that would be obtained by the use of the boundary layer approximation. The leading term of ψ_1, in particular, represents the first order perturbation to the inviscid motion (that is, the flow at infinite Reynolds number). As expected, the speed of the inviscid flow is non-zero at the bounding cylinders being of amount $2\beta\gamma q_2 \cos\phi$ at the inner cylinder and $2\gamma q_1 \cos\phi$ at the outer cylinder.

The behaviour of the stream function $\psi_0+\gamma\psi_1$ outside the boundary layers is typical of more general closed motions. It was to be expected that $\psi_0+\gamma\psi_1$ would be expressible as an asymptotic series of the form $\sum\psi^{(n)}R^{-n/2}$ where the coefficients $\psi^{(n)}$ are independent of R, and further (Batchelor 1956) that the inviscid motion would have uniform vorticity. It now appears that the vorticity associated with the higher order terms of this expansion is also uniform. With the general assumption that the stream function behave at large Reynolds number R like $\sum\psi^{(n)}R^{-n/2}$, where the coefficients $\psi^{(n)}$ are independent of R, this property is true of any two-dimensional motion whose streamlines are closed and lie in a domain free of shear layers. The proof of this result entails only a trivial extension by induction of the argument due to Batchelor, and will not be given here (though it may perhaps be remarked that in the iteration, the fact that the vorticity associated with $\psi^{(n)}$ is uniform follows from the requirement that the pressure distribution associated with $\psi^{(n+2)}$ should be single-valued).

In the present problem the vorticity $\zeta_0+\gamma\zeta_1$ comprises a uniform distribution $2A$ and a distribution which outside the boundary layers becomes transcendentally small (i.e. $o(R^{-n})$, for arbitrarily large n) when the Reynolds number is large. This latter distribution, it will be recalled, divides into a part which decays 'exponentially' rapidly with increasing distance from the inner boundary and a part which decays 'exponentially' rapidly with distance from the outer boundary. Further, each of these parts represents the vorticity distribution, apart only from a constant term and a term which is transcendentally small, in the appropriate boundary layer. By regarding the vorticity distribution as a uniform field with two 'exponentially decaying' fields superposed, we gain a picture of the distribution which comprehends at once both the behaviour of the vorticity in the boundary layers and the result that the vorticity is uniform, apart from a transcendentally small term, outside the boundary layers.

The concept of vorticity decaying so rapidly with increasing distance from the generating surfaces as to become transcendentally small outside a thin layer is familiar, and, in the boundary layer approximation, fundamental. The possible occurrence of a component in the vorticity distribution which does not vary rapidly with distance is less well recognised. The occurrence of such a component is, however, *a priori* to be expected. For any solution of the vorticity equation for which the diffusion term vanishes identically may supply a component which does not change rapidly when the Reynolds number is large. In the case of a two-dimensional motion with closed streamlines whose stream function can be represented as above in the form $\sum \psi^{(n)} R^{-n/2}$ it appears that the *only* possible slowly varying component is one for which the diffusion term vanishes. This is also true of certain axi-symmetric motions with closed streamlines (Batchelor 1956).

Concerning the mode of representation of the stream function ψ_1 it is interesting to note that the asymptotic expansions (34 c), (35), (36) may be embraced in a single representation which holds uniformly throughout the fluid. Such a representation is achieved by writing for ψ_I its asymptotic series (34 c) and for ψ_{R1}, ψ_{R2} their 'boundary layer' representations and then combining all the terms. Thus, we put

$$\begin{aligned}\psi_1 = {} & -\frac{A}{\rho}(\rho^2-1)(\beta^2-\rho^2)\cos\phi + \frac{2\beta}{R^{1/2}(\beta^2-1)} \times \\ & \times \left[\left\{\frac{\beta^2 q_2}{q_1} + \sqrt{\left(\frac{\beta q_1}{q_2}\right)}\right\}\frac{1}{\rho} - \left\{\frac{q_2}{q_1} + \sqrt{\left(\frac{\beta q_1}{q_2}\right)}\right\}\rho\right]\cos\left(\phi - \tfrac{1}{4}\pi\right) + \\ & + \sum_{n=2}^{m} \mathscr{R}\{(l_n\rho + m_n\rho^{-1})\, e^{i\phi}\}R^{-n/2} + \sum_{n=2}^{m} \mathscr{R}\{Q_n\, e^{i\phi-\sqrt{i\xi}}\}R^{-n/2} + \\ & + \sum_{n=2}^{m} \mathscr{R}\{T_n\, e^{i\phi-\sqrt{(iq_2/\beta q_1)\eta}}\}R^{-n/2} + o(R^{-m/2}). \qquad (37)\end{aligned}$$

In the boundary layers, the weakly rotational component may be re-expressed in re-scaled coordinates, and (37) then transforms into (35), (36). Outside the boundary layers, the strongly rotational component is transcendentally small, and (37) then reduces to (34 c). A similar representation holds for the stream function $\psi_0 + \gamma\psi_1$. Both representations correspond with the division of the vorticity field into a uniform component and two 'exponentially decaying' components.

6. The higher order approximations of boundary layer theory

In the following, an expansion for ψ_1 which holds uniformly throughout the flow region is calculated by the methods of boundary layer theory. For simplicity attention is confined to the case in which $A = 0$. The key to the procedure to be used is the knowledge that ψ_1 can be expressed as in (37). Lagerstrom & Cole have proposed that a stream function may generally be expressed (flows with separating boundary layers excepted) as a sum of components of the form $\sum_{n=0}^{\infty} \mathscr{I}_n(z_1, z_2) R^{-n/2}$ and $\sum_{n=0}^{\infty} \mathscr{R}_n(z_1, z_2) R^{-n/2}$, where z_1, z_2 are coordinates such that z_2 vanishes on the solid boundaries, and R is a Reynolds number. The second component is further supposed to be transcendentally small in the region outside the boundary layers. Once it has been established, or can be regarded as established, that the stream function may be expressed in this form it is not difficult to devise a procedure for calculating the higher order corrections to the Prandtl approximation. The aim in what follows is to illustrate this procedure for a motion which is closed.

We begin, then, by assuming that for large R,

$$\psi_1 = \sum_{n=0}^{m} \mathscr{I}_n(\rho, \phi) R^{-n/2} + \sum_{n=1}^{m} \mathscr{R}_{n1}(\xi, \phi) R^{-n/2} + \sum_{n=1}^{m} \mathscr{R}_{n2}(\eta, \phi) R^{-n/2} + o(R^{-m/2}), \tag{38}$$

the terms $R_{n1}(\xi, \phi)$ being supposed transcendentally small for large ξ (i.e. $o(\xi^{-n})$, for any n) and the terms $R_{n2}(\eta, \phi)$ being supposed transcendentally small for large η. The leading summation represents the (truncated) asymptotic series for ψ_1 appropriate to a fixed point in the fluid.

As was remarked in the preceding section, the vorticity associated with the terms $\mathscr{I}_n$ must be uniform throughout the motion. Thus

$$\nabla^2 \mathscr{I}_n = \delta_n, \tag{39}$$

where the δ_n are constants independent of position. To obtain equations for $R_{n1}(\xi, \phi)$ we consider the boundary layer at the inner cylinder. The contribution of the terms $\mathscr{R}_{n2}$ to ψ_1 may here be neglected. On expressing the variables that occur in the equations of motion (13) in terms of ξ, ϕ and then expanding them in powers of $R^{-1/2}$, we get

$$\left.\begin{aligned} &\frac{\partial^4 \mathscr{R}_{11}}{\partial \xi^4} - \frac{1}{\alpha^2}\frac{\partial^3 \mathscr{R}_{11}}{\partial \xi^2 \partial \phi} = 0, \\ &\frac{\partial^4 \mathscr{R}_{21}}{\partial \xi^4} - \frac{1}{\alpha^2}\frac{\partial^3 \mathscr{R}_{21}}{\partial \xi^2 \partial \phi} = -\frac{2}{\alpha}\frac{\partial^3 \mathscr{R}_{11}}{\partial \xi^3} + \frac{1}{\alpha^3}\frac{\partial^2 \mathscr{R}_{11}}{\partial \xi \partial \phi} - \frac{2\xi}{\alpha^3}\frac{\partial^3 \mathscr{R}_{11}}{\partial \xi^2 \partial \phi}, \\ &\cdots\cdots\cdots\cdots \end{aligned}\right\} \tag{40}$$

where $\alpha = 1$. The equations for $\mathscr{R}_{n2}(\eta, \phi)$ may be obtained in a similar way and, in fact, the first two equations are correctly given by (40) if R_{11}, R_{21}, ξ are replaced by R_{12}, R_{22}, $-\eta$ respectively and α is put equal to β. It may be noted that these equations for R_{n1}, R_{n2} contain no terms from the weakly rotational component of ψ_1. This would not be generally true. In general, the weakly rotational component will contribute to the convection velocity in the vorticity equation (though, of course, it will not in general contribute to the vorticity derivatives).

The boundary conditions for the coefficients $\mathscr{I}_n$ of the weakly rotational component are

$$\left.\begin{aligned} &\mathscr{I}_0(1,\phi) = 0, \quad \mathscr{I}_0(\beta,\phi) = \text{constant},\\ &\mathscr{I}_n(1,\phi)+\mathscr{R}_{n1}(0,\phi) = 0\,, \; \mathscr{I}_n(\beta,\phi)+\mathscr{R}_{n2}(0,\phi) = \text{constant} \;\; (n \geqslant 1), \end{aligned}\right\} \quad (41)$$

whilst the boundary conditions for the coefficients $\mathscr{R}_{n1}$, $\mathscr{R}_{n2}$ are

$$\left.\begin{aligned} \frac{\partial \mathscr{R}_{n1}(0,\phi)}{\partial \xi} + \frac{\partial \mathscr{I}_{n-1}(1,\phi)}{\partial \rho} &= 2\cos\phi \quad (n = 0),\\ &= 0 \qquad\quad (n \geqslant 1),\\ \frac{\partial \mathscr{R}_{n2}(0,\phi)}{\partial \eta} - \frac{\partial \mathscr{I}_{n-1}(\beta,\phi)}{\partial \rho} &= 2\cos\phi \quad (n = 0),\\ &= 0 \qquad\quad (n \geqslant 1),\\ \lim_{\xi\to\infty} \mathscr{R}_{n1}(\xi,\phi) = \lim_{\xi\to\infty} \frac{\partial \mathscr{R}_{n1}(\xi,\phi)}{\partial \xi} &= 0,\\ \lim_{\eta\to\infty} \mathscr{R}_{n2}(\eta,\phi) = \lim_{\eta\to\infty} \frac{\partial \mathscr{R}_{n2}(\eta,\phi)}{\partial \eta} &= 0. \end{aligned}\right\} \quad (42)$$

In addition, each of the $\mathscr{I}_n$, $\mathscr{R}_{n1}$ and $\mathscr{R}_{n2}$ and the associated pressure distributions must be periodic functions of ϕ. The conditions on $\mathscr{R}_{n1}$, $\mathscr{R}_{n2}$ at the outer edges of the boundary layers result from supposing that these terms are transcendentally small outside the boundary layers. The boundary conditions for the $\mathscr{I}_n$ derive from the requirement that the normal component of velocity at the boundary associated with these terms should annul the normal component of velocity at the boundary associated with the terms $\mathscr{R}_{n1}$, $\mathscr{R}_{n2}$. The boundary conditions for the $\mathscr{R}_{n1}$, $\mathscr{R}_{n2}$ arise from adjusting the tangential component of velocity associated with the terms $\mathscr{I}_n$ to the tangential velocity of the boundary.

The iterative solution of these equations is complicated by the fact that the equations that govern $\mathscr{I}_n$ ((39) and (41)) are incomplete. That is, if $\mathscr{R}_{n1}$, $\mathscr{R}_{n2}$ are supposed known, $\mathscr{I}_n$ is undetermined to the extent of two constants. One of these constants is δ_n and represents the (uniform) vorticity associated with $\mathscr{I}_n$. The other, Γ_n say, arises because the fluid encircles a closed boundary, and may be defined as the circulation round the inner cylinder of the motion associated with $\mathscr{I}_n$.

In the present problem it is *a priori* obvious from the symmetry of the configuration that δ_n and Γ_n vanish. (This may be seen by considering the motion in which the sign of γ is reversed.) The indeterminancy in $\mathscr{I}_n$ is thus resolved immediately. To resolve the indeterminancy without recourse to a symmetry argument, it is necessary to pass to the equations governing $\mathscr{R}_{n+1,1}$, $\mathscr{R}_{n+1,2}$. On solving for $\mathscr{R}_{n+1,1}$ a condition on the tangential velocity distribution $\partial\mathscr{I}_n(1,\phi)/\partial\rho$ emerges. Similarly, on solving for $\mathscr{R}_{n+1,2}$ a condition emerges on the tangential velocity $\partial\mathscr{I}_n(\beta,\phi)/\partial\rho$. From these two conditions, we may then deduce that $\delta_n = 0$, $\Gamma_n = 0$, and so on.

The equations for $\mathscr{I}_n$, $\mathscr{R}_{n1}$, $\mathscr{R}_{n2}$ may otherwise be solved without further difficulty. For the leading terms we get

$$\left.\begin{aligned}
\mathscr{I}_0 &= 0, \qquad \mathscr{I}_1 = \frac{2}{\beta^2-1}\left[\frac{2\beta^2}{\rho} - (\beta^2+1)\rho\right]\cos(\phi - \tfrac{1}{4}\pi),\\
\mathscr{I}_2 &= \frac{2}{(\beta^2-1)^2}\left[\frac{4\beta^2(\beta^2+1)}{\rho} + (\beta^4 - 10\beta^2+1)\rho\right]\sin\phi,\\
\mathscr{R}_{11} &= -2\mathscr{R}\left\{\frac{1}{\sqrt{i}}e^{-\sqrt{i}\xi + i\phi}\right\},\\
\mathscr{R}_{21} &= -2\mathscr{R}\left\{\left(\frac{\xi^2}{2} + \frac{2\xi}{\sqrt{i}} + \frac{5\beta^2-1}{(\beta^2-1)i}\right)e^{-\sqrt{i}\xi + i\phi}\right\},\\
\mathscr{R}_{12} &= 2\beta\mathscr{R}\left\{\frac{1}{\sqrt{i}}e^{-\sqrt{i}(\eta/\beta) + i\phi}\right\},\\
\mathscr{R}_{22} &= -2\mathscr{R}\left\{\left(\frac{\eta^2}{2\beta} + \frac{2\eta}{\sqrt{i}} + \frac{\beta(\beta^2-5)}{(\beta^2-1)i}\right)e^{-\sqrt{i}(\eta/\beta) + i\phi}\right\}.
\end{aligned}\right\} \qquad (43)$$

when these terms are combined together as in (38) they yield an expression for ψ_1 which holds to $O(R^{-1})$ uniformly throughout the fluid.

7. The terms of higher order in γ

For the sake of completeness it is worth noting that the higher order terms of the perturbation of the motion may be solved in essentially the same way as the first order terms.

When the equations governing the higher order terms ζ_n, ψ_n of the expansions for the vorticity component ζ and stream function ψ are examined, it is seen that each varies with ϕ according to the respective forms

$$\zeta_n = \sum_{s=0}^{n} g_{n,\,n-2s}\, e^{i(n-2s)\phi}, \qquad \psi_n = \sum_{s=0}^{n} f_{n,\,n-2s}\, e^{i(n-2s)\phi}. \qquad (44)$$

The equations governing $g_{n,s}$, $f_{n,s}$ are then

$$g''_{n,s} + \frac{1}{\rho}g'_{n,s} - \left(RAsi + \frac{RBsi + s^2}{\rho^2}\right)g_{n,s} = RG_{n,s}, \qquad (45\text{ a})$$

$$f''_{n,s} + \frac{1}{\rho}f_{n,s} - \frac{s^2}{\rho^2}f_{n,s} = -g_{n,s} + F_{n,s}, \qquad (45\text{ b})$$

where $G_{n,s}$, $F_{n,s}$ are defined by

$$\left.\begin{aligned}
\sum_{s=0}^{n} G_{n,\,n-2s}(\rho)e^{i(n-2s)\phi} &= \frac{1}{\rho}\sum_{s=1}^{n-1}\left(\frac{\partial\psi_r}{\partial\phi}\frac{\partial\zeta_{n-r}}{\partial\rho} - \frac{\partial\psi_r}{\partial\rho}\frac{\partial\zeta_{n-r}}{\partial\phi}\right),\\
\sum_{s=0}^{n} F_{n,\,n-2s}(\rho)e^{i(n-2s)\phi} &= 4\rho\cos\phi\nabla^2\psi_{n-1} + 2\rho^2(2+\cos 2\phi)\nabla^2\psi_{n-2} +\\
&\qquad + 4\rho^3\cos\phi\nabla^2\psi_{n-3} + \rho^4\nabla^2\psi_{n-4}.
\end{aligned}\right\} \qquad (46)$$

Further the boundary conditions are

$$\left.\begin{aligned}
&f_{n,s}(1) = 0, \quad f_{n,0}(\beta) = \text{constant}, \quad f_{n,s}(\beta) = 0 \quad (s \neq 0),\\
&f'_{n,s}(1) = (-1)^{n-1}, \quad f'_{n,s}(\beta) = -(-\beta)^n\,(q_1/q_2), \quad (s = \pm n),\\
&f'_{n,s}(1) = 0, \quad f'_{n,s}(\beta) = -(-\beta)^{n-2}(\beta^2-1) \quad (|s| < n),
\end{aligned}\right\} \qquad (47)$$

together with the condition that each term of the expression for the pressure is singled-valued.

These equations for the component functions $f_{n,s}(\rho)$ of the higher order perturbations ψ_n are similar to the equations for the component term $f(\rho)$ of the first order stream function ψ_1, and may be solved in a similar manner. The only features which call for special treatment are the interaction terms $F_{n,s}(\rho)$ and $G_{n,s}(\rho)$ and the components of the vorticity that do not vary with ϕ. The interaction terms in the equations for $f_{n,s}$ depend on the components $f_{r,s}$ for $r = 1, 2 \dots n-1$, and are determined iteratively. To determine the components of the vorticity that do not vary with ϕ, it is necessary to use the pressure condition. It is found on examining the momentum equations that the necessary and sufficient condition for the pressure to be single-valued is that

$$g'_{n,0} = \frac{R}{\rho}\int_1^\rho \rho G_{n,0}\, d\rho,$$

whence

$$g_{n,0} = \text{constant} + R\int_1^\rho \frac{1}{\rho}\int_1^{\rho'} \rho' G_{n,0}(\rho')\, d\rho' d\rho. \tag{48}$$

The corresponding term of the stream function, namely $f_{n,0}$, may now be determined in the same way as the other terms $f_{n,s}$, $s \neq 0$.

It is not proposed to investigate these equations further. Enough has been set down to show that the solution may be completed in essentially the same way as for the first order terms. (The homogeneous form of (45 a) now has the general solution $2CJ_{\mu^*}(z^*) + 2DH^1_{\mu^*}(z^*)$, where $\mu^* = \sqrt{(-RAsi,)}$ $z^* = \sqrt{(RBsi+1)}$ and $A \neq 0$). Moreover the solution may be expected to behave at large Reynolds numbers in much the same way as did the first order terms. In particular it may be tentatively inferred that for large R, the vorticity may be expressed as the sum of a constant term $\sum_{n=0}^{\infty} g_{2n,0}\,\gamma^{2n}$, and an infinity of terms which vary exponentially rapidly with R. It may also be tentatively inferred that the stream functions ψ_n may at large Reynolds numbers be expressed as in (38). If this proves true, the expansions of ψ, ζ in powers of γ may be expected to converge uniformly with respect to R, when γ is sufficiently small.

The author is grateful to Dr I. Proudman and Dr H. C. Levey for many helpful criticisms of this paper. Acknowledgment is also made to the Chief Scientist, Department of Supply, for permission to publish.

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